1 The Hats Problem

The original problem is usually posed as follows: Eight professors walk into a restaurant, each wearing hats. They hang their hats, and sit down to eat. As they walk out, each one picks a random hat, and wears it. What is the probability that each of the professors wears a hat not belonging to himself?

2 First Observations

The first observation is that there are a total of $8! = 40320$ possible permutations of the eight hats. One method for solving the problem would be to list of the permutations, count the ones where no hat occurs at the correct position, and just take the ratio. This brute-force approach is possible (especially when you can use a computer to do it), but is somehow unsatisfactory.

The second observation is that if we do not enumerate the possibilities, there is no "easy" way out by considering the process. Number the hats and professors from 1 to 8. Now it is obvious that the first professor has a $7/8$ probability of picking an incorrect hat. Things immediately start getting messy with the second professor. There are two cases, if the first professor took hat number 2, the second professor will definitely get an incorrect hat. If the first professor picked someone else's hat, then the second professor will pick an incorrect hat with probability $6/7$. The tree grows with each extra professor considered.

This "tree" method can also be used to solve the problem, but the calculation is, again, messy. Once again, a computer can trivialize this process. But we will not stop here.

3 An Alternative Approach

The alternative point of view is to look at the solution to this problem as a special case of a more general problem: Consider n professors with n hats, picking hats randomly. What is the probability that k of them end up wearing incorrect hats?

To this end, let us define

$T(n, k)$: The probability that k professors wear incorrect hats, given there are n professors with n hats ($k \leq n$).

Let us see how things work with small numbers. $T(0, 0)$, which is the probability of zero incorrect hats, given zero hats, is obviously 1:

$$T(0,0) = 1$$

How about $T(1, 0)$, and $T(1, 1)$? With one person and one hat, you can not possibly make a mistake. Therefore:

$$\begin{aligned} T(1,0) &= 1 \\ T(1,1) &= 0 \end{aligned}$$

With two professors, there are two permutations of the hats. One with both correct, and one with both incorrect. So, we get:

$$\begin{aligned} T(2,0) &= 1/2 \\ T(2,1) &= 0 \\ T(2,2) &= 1/2 \end{aligned}$$

With three professors, we have the following six permutations:

1	**2**	**3**
1	3	2
2	1	**3**
2	3	1
3	1	2
3	**2**	1

The hats which are in correct places are in boldface numerals. There is one permutation with zero incorrect hats, no permutations with one incorrect hat, three permutations with two incorrect hats, and two permutations with three incorrect hats. Thus, we now have:

$$\begin{aligned} T(3,0) &= 1/6 \\ T(3,1) &= 0 \\ T(3,2) &= 1/2 \\ T(3,3) &= 1/3 \end{aligned}$$

We will go one more step, and work out the case with four professors. Now we have 24 permutations, which you can see below. Again, the hats which are in their correct places are in boldface numerals.

By inspection, we can see the following:

Number of incorrect hats	Number of permutations
0	1
1	0
2	4
3	8
4	9

Now we have that:

$$\begin{aligned} T(4,0) &= 1/24 \\ T(4,1) &= 0 \\ T(4,2) &= 1/6 \\ T(4,3) &= 1/3 \\ T(4,4) &= 3/8 \end{aligned}$$

1	**2**	**3**	**4**
1	**2**	4	3
1	3	2	**4**
1	3	4	2
1	4	2	3
1	4	**3**	2
2	1	**3**	**4**
2	1	4	3
2	3	1	**4**
2	3	4	1
2	4	1	3
2	4	**3**	1
3	1	2	**4**
3	1	4	2
3	**2**	1	**4**
3	**2**	4	1
3	4	1	2
3	4	2	1
4	1	2	3
4	1	**3**	2
4	**2**	1	3
4	**2**	**3**	1
4	3	1	2
4	3	2	1

That is enough manual labor to get a feeling for the figures involved in the problem. Now our attack will be more on the mathematical side. Let us first present the results we have found for $T(n,k)$ so far in tabular form:

$T(n,k)$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
$n=0$	1				
$n=1$	1	0			
$n=2$	$1/2$	0	$1/2$		
$n=3$	$1/6$	0	$1/2$	$1/3$	
$n=4$	$1/24$	0	$1/6$	$1/3$	$3/8$

4 The Mathematical Attack

Now we can try to attack the problem in a more mathematical way. We need some observations on $T(n,k)$ to proceed in the solution. The first observation is the following:

$$T(n,0) = \frac{1}{n!}$$

Since $T(n,0)$ stands for the probability that given there are n professors and n hats, no hats are chosen incorrectly, it is simply the probability that every professor gets his own hat. There is only one permutation where every professor gets his own hat, so the probability is just one over the total number of permutations.

Since the total probability must be normalized to one with any given number of hats, the following condition is also true:

$$\sum_{k=0}^{n} T(n,k) = 1$$

Now comes the key observation to our solution. In general, $T(n,k)$ is the probability that given n hats and professors, k of them wear incorrect hats. We claim that:

$$T(n,k) \quad = \quad \frac{1}{(n-k)!}T(k,k)$$

This is true because of the following fact. Having k incorrect hats means that we have $n-k$ hats that are worn correctly. The two groups of professors, namely those that wear their own hats, and those that wear someone else's hat, are mutually exclusive. Thus, the event that k professors wear incorrect hats, and the event that $n-k$ professors wear correct hats, are independent. Thus, the probability splits into a product: The probability that $n-k$ professors wear their own hats and the probability that k professors all wear incorrect hats. The result is the above equation.

Note that the answer to the original problem is now $T(8,8)$. But, our equations do not yet give us an easy way out. For instance, applying the last formula to $T(8,8)$, we get:

$$T(8,8) \quad = \quad \frac{1}{0!}T(8,8)$$

Just an identity. However, once we know, for example, $T(4,4)$, the following are immediately known:

$$\begin{aligned}
T(5,4) &= \frac{1}{1!}T(4,4) \\
T(6,4) &= \frac{1}{2!}T(4,4) \\
T(7,4) &= \frac{1}{3!}T(4,4)
\end{aligned}$$

So, the real trouble is finding $T(k,k)$. Let us define:

$$T(k) \quad = \quad T(k,k)$$

Now our problem is reduced to calculating $T(k)$.

5 Formulating the Problem

Using the equations

$$\sum_{k=0}^{n} T(n,k) = 1$$

and

$$T(n,k) = \frac{1}{(n-k)!} T(k)$$

we can write the following equation:

$$\sum_{k=0}^{n} \frac{T(k)}{(n-k)!} = 1$$

This is, in fact, an infinite number of equations. The first few equations read:

$$\begin{aligned}
T(0) &= 1 \\
T(0) + T(1) &= 1 \\
\frac{T(0)}{2!} + T(1) + T(2) &= 1 \\
\frac{T(0)}{3!} + \frac{T(1)}{2!} + T(2) + T(3) &= 1 \\
\frac{T(0)}{4!} + \frac{T(1)}{3!} + \frac{T(2)}{2!} + T(3) + T(4) &= 1
\end{aligned}$$

These equations can be solved to obtain all $T(k)$. It is easily verified that solving these equations gives us the same results that we have calculated in the table.

6 The Solution

The solution for $T(k)$ is difficult to obtain from the equation directly. However, extending the table a little bit and looking at the numbers and how they develop, makes the following solution apparent:

$$T(k) = \sum_{m=0}^{k} \frac{(-1)^m}{m!}$$

It is easy to verify that the solution matches our table:

$$\begin{aligned}
T(0) = 1/0! &= 1 \\
T(1) = 1/0! - 1/1! &= 0 \\
T(2) = 1/0! - 1/1! + 1/2! &= 1/2 \\
T(3) = 1/0! - 1/1! + 1/2! - 1/3! &= 1/6 \\
T(4) = 1/0! - 1/1! + 1/2! - 1/3! + 1/4! &= 3/8
\end{aligned}$$

But of course this is not proof that it is the solution for all k. To complete the proof, we need to show that this expression indeed satisfies the equation for $T(k)$.

7 The Proof

We will now substitute

$$T(k) = \sum_{m=0}^{k} \frac{(-1)^m}{m!}$$

into

$$\sum_{k=0}^{n} \frac{T(k)}{(n-k)!} = 1$$

and try to show that it indeed satisfies the equation.

Simple substitution yields:

$$\sum_{k=0}^{n} \sum_{m=0}^{k} \frac{(-1)^m}{(n-k)!m!} = 1$$

How will we prove this to be true? If we are correct, then this must turn out to be an identity.

What we have is a two dimensional sum. The region of summation is a triangle, as can be seen in figure 1.

Our trick is going to be the following: We will change the order of summation, so that we sum along the *diagonal* first. The formula for the diagonals is $k = m+s$, where s will take values from 0 to n. We will replace k with $m+s$. Then the summation over m will be from 0 to $n-s$. Then, the summation over s will be done, from 0 to n. The resulting double summation is as follows:

$$\sum_{s=0}^{n} \sum_{m=0}^{n-s} \frac{(-1)^m}{(n-s-m)!m!} = 1$$

Supposedly this made things better. But how? In order to appreciate this, we need one more side trip.

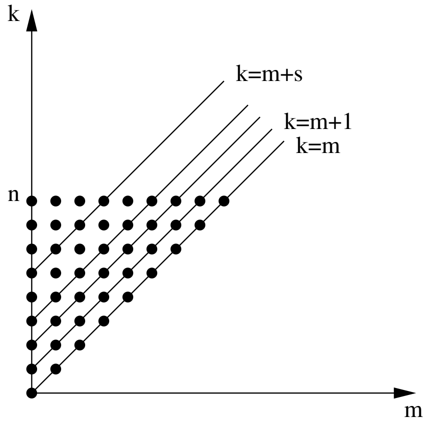

Figure 1: Region of summation.

Recalling The Binomial Theorem

The binomial theorem states that, for any a, b, and integer n:

$$(a+b)^n = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} a^k b^{n-k}$$

Let us apply this to the special case where $a = -1$, and $b = 1$. We get the following:

$$(1-1)^n = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} (-1)^k 1^{n-k}$$

which means

$$\sum_{k=0}^{n} \frac{n!}{(n-k)!k!} (-1)^k = 0^n$$

Since $n!$ is independent of k, we finally have:

$$\sum_{k=0}^{n} \frac{(-1)^k}{(n-k)!k!} = 0^n$$

We have kept the 0^n rather than putting in just 0, because there is a fine point here. As long as $n > 0$, the sum we have is indeed zero:

$$\sum_{k=0}^{n} \frac{(-1)^k}{(n-k)!k!} = 0, \quad n > 0$$

But when $n = 0$, there is only one term which is 1, thus the sum is 1:

$$\sum_{k=0}^{n} \frac{(-1)^k}{(n-k)!k!} = 1, \quad n = 0$$

Putting it Together

Going back to the last equation, we have:

$$\sum_{s=0}^{n} \sum_{m=0}^{n-s} \frac{(-1)^m}{(n-s-m)!m!} = 1$$

Now, looking at the inner sum, we can see that the summation over m is just the expansion of $(1-1)^{n-s}$, which is zero as long as $n-s$ is non-zero. So, the only surviving term is the term where $s = n$, and that is exactly one. This completes the proof that this last equation is an identity.

The Conclusion, and Final Observations

Our final, most general answer is:

$$T(n,k) = \frac{1}{(n-k)!} \sum_{m=0}^{k} \frac{(-1)^m}{m!}$$

which is the probability that with n professors picking from n hats at random, k of them will be wearing someone else's hat.

One important point is to notice that

$$e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!}$$
$$\frac{1}{e} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!}$$

So, the sum included in the answer (which is $T(k)$) is just a truncated series expansion for $1/e$. This is a very rapidly converging series, and with 8 terms or more, the error is less than one part in a thousand. So, we can say:

$$T(n,k) = \frac{1}{(n-k)!} \frac{1}{e}, \quad k \geq 8$$

Let us also not forget the answer to the original problem:

$$
\begin{aligned}
T(8,8) &= T(8) = 1/0! - 1/1! + 1/2! - 1/3! + 1/4! - 1/5! + 1/6! - 1/7! + 1/8! \\
&= 14833/40320 \\
&= 2119/5760 \\
&= 0.36788194
\end{aligned}
$$

If our last claim is true, this must be very close to $1/e$:

$$\frac{1}{e} = 0.36787944$$

Perhaps the most interesting result is that, the answer would be almost the same if we had twenty, a hundred, or billions of professors. The probability that *everyone* gets the wrong hat converges very rapidly to $1/e$, and it stays there!

Lastly, there is one final point. Suppose you are not quite happy with the approximate result of $1/e$ for the problem with n professors. You want to find the exact fraction. According to our answer, you need to add the alternating inverse factorial series. But, there is an easier way if you have an accurate value for e at hand. By some theorem, when adding an infinite alternating series, the error is less than the first term that is left out in magnitude. This implies that (if you put some thought into it), the actual correct fraction can be found as described below:

Let n be the number of professors. In this case, there are $n!$ permutations of the hats in total. Therefore, our answer can be expressed as:

$$T(n) = \frac{P(n)}{n!}$$

where $P(n)$ is the number of permutations where every professor wears an incorrect hat. Because the error in approximating this fraction with $1/e$ is less than $1/(n+1)!$ in magnitude, we have the following relation:

$$P(n) = \left\lfloor \frac{n!}{e} + \frac{1}{2} \right\rfloor$$

that is, $P(n)$ is $n!/e$ rounded to the nearest integer.

With 8 professors, the number of permutations is $8! = 40320$. $40320/e = 14832.899$, thus rounding to the nearest integer, we have that:

$$T(n) = \frac{14833}{40320}$$

which we know to be the correct answer.