1 The Harmonic Oscillator

The typical harmonic oscillator is the mass-on-a-spring system, which is described by the following equation:

$$m\ddot{x} + b\dot{x} + kx = 0 \tag{1}$$

where m is the mass, k is the spring constant, and b is the coefficient of the "viscous" damping term, which represents a force proportional to the speed of the mass. (And x, of course, represent the displacement of the mass from its equilibrium point.

1.1 The Free Harmonic Oscillator

We can have a simpler case, by setting the damping term to zero. Then we have the even simpler equation:

$$m\ddot{x} + kx = 0 \tag{2}$$

The solution to this case is well-known, and it is sinusoidal motion:

$$x = A\cos\omega_0 t + \phi \tag{3}$$

where $\omega_0 = \sqrt{k/m}$ and A and ϕ are constants to be determined by the initial conditions. The constant ω_0 is called the natural frequency of the system since it is the frequency it will osciallte when it is left "alone".

1.2 The Damped Harmonic Oscillator

Going back to Equation 1, let us find the solution for the damped case. First, let us divide through by m to obtain:

$$\ddot{x} + b/m\dot{x} + k/mx = 0$$

Now, by substituting ω_0^2 for k/m and $2z$ for b/m we obtain (the reason for the coefficient of 2 will be clear soon):

$$\ddot{x} + 2z\dot{x} + \omega_0^2 x = 0 \tag{4}$$

Substituting a solution of the form $x = e^{st}$, we get the equation:

$$s^2 e^{st} + 2zse^{st} + \omega_0^2 e^{st} = 0 \tag{5}$$

from which we find:

$$s = -z \pm \sqrt{z^2 - \omega_0^2} \tag{6}$$

This simply means that our solution is:

$$\begin{aligned} x &= Ae^{s_1 t} + Be^{s_2 t} \\ s_1 &= -z + \sqrt{z^2 - \omega_0^2} \\ s_2 &= -z - \sqrt{z^2 - \omega_0^2} \end{aligned} \tag{7}$$

This is always the form of the solution, but the quality varies with the relative magnitudes of z and ω_0. (We are only interested in cases where $z > 0$.) We can distingush three cases, which we will look at:

1.2.1 The Case $z < \omega_0$

In this case, the inside of the radical is negative. Thus, after a little work, we can find that:

$$x = A_d e^{-zt} \cos(\omega_d t + \phi_d) \tag{8}$$
$$\omega_d = \sqrt{\omega_0^2 - z^2}$$

Once again, the constants A_d and ϕ_d are to be determined by the initial conditions. The motion is still sinusoidal, but its envelope is an exponential. The angular frequency of the motion is now modified, and is ω_d instead of ω_0. Note that $\omega_d < \omega_0$. This is usually referred to as the *underdamped case*.

1.2.2 The Case $z > \omega_0$

Here, the inside of the radical is positive, so we have two distinct, real roots. The form of the solution then becomes:

$$x = A_1 e^{-\lambda_1 t} + A_2 e^{-\lambda_2 t} \tag{9}$$
$$\lambda_1 = z + \sqrt{z^2 - \omega_0^2}$$
$$\lambda_2 = z - \sqrt{z^2 - \omega_0^2}$$

The constants A_1 and A_2 are determined by the initial conditions. So, there is no osciallation at all, and the motion dies off exponentially. This is known as the *overdamped case*.

1.2.3 The Case $z = \omega_0$

In this case, the radical is zero, and we have two identical roots. The form of the solution then becomes:

$$x = (A_c + B_c t)e^{-zt} \tag{10}$$

This is the case where motion dies off the quickest, and there is no oscillation. The constants A_c and B_c are determined by the initial conditions. This is the *critically damped* case.

2 The Forced Harmonic Oscillator

When the harmonic oscillator is forced to motion by a sinusoidal driving force, we have the following equation of motion:

$$m\ddot{x} + b\dot{x} + kx = F \cos \omega t \tag{11}$$

Here, F is the maximum force, and ω is the driving frequency. Dividing through by m again, we obtain:

$$\ddot{x} + 2z\dot{x} + \omega_0^2 x = (F/m) \cdot \cos \omega t \tag{12}$$

where the definitions are as before.

We are really interested in the steady-state solution, since we know that the homogenous solutions die off (given there is damping). The easier way of going about this is using a complex exponential rather than a cosine for the force term, and keep in mind that the *real* part of the solution has physical significance. So, our complex equation is:

$$\ddot{x_c} + 2z\dot{x_c} + \omega_0^2 x_c = (F/m) \cdot e^{i\omega t} \tag{13}$$

A particular solution can be found by using a solution of the form $x_c = Ae^{i\omega t}$. Substituting, we find:

$$-\omega^2 A + 2i\omega z A + \omega_0^2 A = F/m \tag{14}$$

which yields

$$A = \frac{F/m}{\omega_0^2 - \omega^2 + 2i\omega z} \tag{15}$$

Thus the solution is:

$$x_c = \frac{(F/m) \cdot e^{i\omega t}}{\omega_0^2 - \omega^2 + 2i\omega z} \tag{16}$$

And of course, the "real" solution is:

$$x = \Re\{x_c\} \tag{17}$$

2.1 Power Dissipation Spectrum

Obviously, the damped harmonic oscillator oscillates with the driving frequency. The amplitude of oscillation is a constant that depends on the driving frequency. Thus, the total energy content of the oscillator is fixed. However, energy is being dissipated by the damping term, which is in turn being supplied by the driving force. The question is, what is the average power dissipation per time, and how does its spectrum change with the driving frequency? We can find the answer as follows. Power is just the force multiplied by the velocity, and the average power is[1]:

$$\overline{P} = \frac{1}{2} \cdot \Re\{F \cdot (i\omega A)^*\} = \Re\{F^* \cdot i\omega A\} \tag{18}$$

Thus:

$$\overline{P} = \frac{1}{2} \cdot \Re\left\{ \frac{(F^2/m) \cdot i\omega}{\omega_0^2 - \omega^2 + 2i\omega z} \right\} \tag{19}$$

After some manipulation, and taking the real part, we find:

$$\overline{P} = \frac{F^2}{4mz} \cdot \frac{4z^2\omega^2}{(\omega_0^2 - \omega^2)^2 + 4z^2\omega^2} \tag{20}$$

[1] See the appendix for what this means.

The interesting part about Equation 20 is that the second fraction is of the form $b^2/(a^2+b^2)$. Thus, the maxiumum value of that fraction is one, and the minimum value is zero. The maxiumum occurs when $\omega = \omega_0$. So we can state that:

$$\overline{P}_{\max} = \frac{F^2}{4mz} \tag{21}$$

Another interesting and often referenced quantity the the "full width at half-maximum" of the power spectrum. In other words, we want to find the two driving frequencies where the power is at one-half its maximum value, and we want to find the difference of the two angular frequencies.

Our equation is then:

$$\overline{P} = \overline{P}_{\max}/2 \tag{22}$$

Just by inspection, we can figure out that this happens if and only if:

$$(\omega_0^2 - \omega^2)^2 = 4z^2\omega^2 \tag{23}$$

which means

$$\omega_0^2 - \omega^2 = \pm 2z\omega \tag{24}$$

and finally leads to the quadratic equation:

$$\omega^2 \pm 2z\omega - \omega_0^2 = 0 \tag{25}$$

This gives us a total of *four* solutions:

$$\begin{aligned}
\omega_1 &= \sqrt{z^2+\omega_0^2} - z \\
\omega_2 &= \sqrt{z^2+\omega_0^2} + z \\
\omega_3 &= -\sqrt{z^2+\omega_0^2} - z \\
\omega_4 &= -\sqrt{z^2+\omega_0^2} + z
\end{aligned}$$

However, of the four solutions, ω_3 and ω_4 are negative. So, the really interesting solutions are ω_1 and ω_2. The full-width at half-maximum (FWHM) is thus:

$$\text{FWHM} = \omega_2 - \omega_1 = 2z \tag{26}$$

So, the FWHM of the power spectrum is exactly 2z, which is also b/m in terms of the original variables. Note that the half-power points are not symmetric around ω_0.

Another quantity of interest is the so-called "quality factor" Q. There a few different definitions, but in our case, the definition that is easier to use is:

$$Q = \frac{\text{Peak frequency}}{\text{FWHM}} \tag{27}$$

Substituting our findings, we find that:

$$Q = \frac{\omega_0}{2z} = \frac{\sqrt{km}}{b} \tag{28}$$

2.2 Power Dissipation With No Damping

$$\overline{P} = \frac{F^2}{4mz} \cdot \frac{4z^2\omega^2}{(\omega_0^2 - \omega^2)^2 + 4z^2\omega^2}$$

Looking back at Equation 20 (which is repeated above for convenience) let us try to see what happens when $b = z = 0$, i.e., there is no damping. We can distinguish two cases. The first case is when $\omega \neq \omega_0$. In this case, $\overline{P} = 0$. This makes sense, since then oscillation proceeds with constant amplitude, there is no dissipation (no friction term), hence there is no power input.

The second case is when $\omega = \omega_0$. In this case, you can see by inspection that $\overline{P}$ diverges! You can not talk about average power. The reason is, when there is no damping, the oscillation amplitude grows with time, and average power keeps increasing without bound. At the same time, the FWHM is zero.

3 Appendix: Calculating Power

Suppose we have a physical quantity that is time dependent, and is the product of other two, time dependent quantities. Let us call it $P(t)$, and the two other quantities $A(t)$ and $B(t)$. Then our equation is:

$$P(t) = A(t)B(t) \tag{29}$$

Let us further assume that $A(t)$ and $B(t)$ vary sinusoidally in time:

$$\begin{aligned} A(t) &= A\cos(\omega t + \phi_A) \\ B(t) &= B\cos(\omega t + \phi_B) \end{aligned} \tag{30}$$

Then, we have:

$$P(t) = AB\cos(\omega t + \phi_A)\cos(\omega t + \phi_B) \tag{31}$$

Using trigonometric identities, we can write this as:

$$P(t) = \frac{1}{2}AB\cos(\phi_A - \phi_B) + \frac{1}{2}AB\cos(2\omega t + \phi_A + \phi_B) \tag{32}$$

The interesting quantity is the average power (averaged over many periods), and since the second term is oscillatory and averages to zero, we get:

$$\overline{P} = \frac{1}{2}\cos(\phi_A - \phi_B) \tag{33}$$

Now, this is all fine. But, sometimes the quantities $A(t)$ and $B(t)$ are not calculated directly, but in complex form such that:

$$\begin{aligned} A(t) &= \Re\{A_c(t)\} \\ B(t) &= \Re\{B_c(t)\} \end{aligned} \tag{34}$$

What can *not* be done is to say that:

$$P(t) = \Re\{A_c(t)B_c(t)\} \text{ (wrong!)} \tag{35}$$

since the real-part operation will not distribute over the product.

To correspond to the actual $A(t)$ and $B(t)$, $A_c(t)$ and $B_c(t)$ must be of the form:

$$\begin{aligned} A_c(t) &= Ae^{i\omega t + i\phi_A} \\ B_c(t) &= Be^{i\omega t + i\phi_B} \end{aligned} \tag{36}$$

How can we obtain the *average real power* without taking the real parts, multiplying, and averaging? One answer is the following:

$$\overline{P} = \frac{1}{2}\Re\{A_c(t)B_c^*(t)\} = \frac{1}{2}\Re\{A_c^*(t)B_c(t)\} \tag{37}$$

Let us verify that this indeed gives the correct answer. Plugging in the values, we get:

$$\begin{aligned} \overline{P} &= \frac{1}{2}\Re\{Ae^{i\omega t + i\phi_A}Be^{-i\omega t - i\phi_B}\} \\ \overline{P} &= \frac{1}{2}\Re\{ABe^{i(\phi_A - \phi_B)}\} \\ \overline{P} &= \frac{1}{2}AB\cos(\phi_A - \phi_B) \end{aligned} \tag{38}$$

So, this approach gives the same result as Equation 33. Thus, we have our simple rule:

If $A(t) = \Re\{A_c(t)\}$ and $B(t) = \Re\{B_c(t)\}$, both of which vary sinusoidally with the same frequency, then we have:

$$\overline{P} = \overline{A(t)B(t)} = \frac{1}{2}\Re\{A_c(t)B_c^*(t)\} \tag{39}$$