PHYS 486 Problem Set #3

1 A Miracle Occurs...

Our goal here is to prove that the following equation is correct:

$$\sum_{x=0}^{2^n-1}(-1)^{(a\cdot x)}(-1)^{y\cdot x} = \prod_{j=1}^{n}\sum_{x_j=0}^{1}(-1)^{(a_j+y_j)x_j}$$

In general, note that the sum over all values of a certain n-bit number may be replaced by n sums making up all possible combinations of its bits, zero and one. In mathematical terms, this means that:

$$\sum_{x=0}^{2^n-1} \iff \sum_{x_1=0}^{1}\sum_{x_2=0}^{1}\sum_{x_3=0}^{1}\cdots\sum_{x_n=0}^{1}$$

Of course, one would have to express any references to x in the sum by references to its bits. So, for instance, if you had x in the sum, you would have to replace it by $\sum_{j=1}^{n} x_j 2^j$. (We will not have to do that here.)

Using the definition of the "number dot product", show that:

$$\sum_{x=0}^{2^n-1}(-1)^{(a\cdot x)}(-1)^{y\cdot x} = \sum_{x_1=0}^{1}\sum_{x_2=0}^{1}\sum_{x_3=0}^{1}\cdots\sum_{x_n=0}^{1}(-1)^{\sum_{k=1}^{n}(a_k+y_k)x_k}$$

By the law of exponents, this can be transformed into:

$$\sum_{x=0}^{2^n-1}(-1)^{(a\cdot x)}(-1)^{y\cdot x} = \sum_{x_1=0}^{1}\sum_{x_2=0}^{1}\sum_{x_3=0}^{1}\cdots\sum_{x_n=0}^{1}\prod_{k=1}^{n}(-1)^{(a_k+y_k)x_k}$$

Observe that the product is exactly that; n terms being multiplied together. Since the first term depends only on x_1, it can be moved out of the first $n-1$ sums, all the way to the left:

$$\sum_{x=0}^{2^n-1}(-1)^{(a\cdot x)}(-1)^{y\cdot x} = \sum_{x_1=0}^{1}(-1)^{(a_1+y_1)x_1}\sum_{x_2=0}^{1}\sum_{x_3=0}^{1}\cdots\sum_{x_n=0}^{1}\prod_{k=2}^{n}(-1)^{(a_k+y_k)x_k}$$

Show that, by repeating this process, one can arrive at the following "product of sums":

$$\sum_{x=0}^{2^n-1}(-1)^{(a\cdot x)}(-1)^{y\cdot x} = \sum_{x_1=0}^{1}(-1)^{(a_1+y_1)x_1}\sum_{x_2=0}^{1}(-1)^{(a_2+y_2)x_2}\cdots\sum_{x_n=0}^{1}(-1)^{(a_n+y_n)x_n}$$

And finally, show that:

$$\sum_{x=0}^{2^n-1}(-1)^{(a\cdot x)}(-1)^{y\cdot x} = \prod_{j=1}^{n}\sum_{x_j=0}^{1}(-1)^{(a_j+y_j)x_j}$$

2 Measuring States

This question considers the answer to question II, part (b) of assignment #3 by David Mermin.

In general, if you have a state $|\psi\rangle$, which you know must be one of $|\alpha\rangle$ or $|\beta\rangle$, the only way you can figure out which one it is is to measure it. The outcome of the measurement will be one of the components of $|\alpha\rangle$ and $|\beta\rangle$. But, if the outcome is contained in both vectors, you will not be able to tell them apart with certainty. This means that, if you are to definitely tell which of the two vectors $|\psi\rangle$ is, the two must have no components in common. So we deduce that $|\alpha\rangle$ or $|\beta\rangle$ are orthogonal, which means $\langle\beta|\alpha\rangle = 0$.

For this problem, let $|\alpha\rangle$ be the state when $f(0) = f(1)$ and let $|\beta\rangle$ be the state when $f(0) \neq f(1)$.

a. Show that $\langle\beta|\alpha\rangle \neq 0$, and find its value.

b. Show that, applying a unitary transformation does not change the value of the inner product of two states, i.e., the value of $\langle\beta|\alpha\rangle$ is unchanged when both states are transformed by the same U. (Remember, $\langle\psi| = (|\psi\rangle)^\dagger$.)

c. By the result of part b, deduce that it is not possible to make the two states orthogonal by unitary transformations since they are not orthogonal in the first place.