PHYS 486 Problem Set #2

1 The Swap Operator Yet Again

The swap operator is given as:

$$S_{ij} = n_i n_j + \overline{n}_i \overline{n}_j + X_i X_j (n_i \overline{n}_j + \overline{n}_i n_j)$$

a. Show that, by expressing the n and $\overline{n}$ operators in terms of Z operators, S_{ij} can be expressed as:

$$\frac{1}{2}(1 + Z_i Z_j) + \frac{1}{2}(X_i X_j)(1 - Z_i Z_j)$$

b.

Now, let us define an additional operator, Y, as $Y = ZX = -XZ$. Show that, with this definition of Y, the swap operator takes the almost symmetric form below:

$$S_{ij} = \frac{1}{2}(1 + X_i X_j - Y_i Y_j + Z_i Z_j)$$

c. Show that with the definitions given in class, the matrix representaion of Y is:

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

2 The Pauli Matrices

The Pauli matrices, much beloved by physicists, are defined as follows (and correspond to our X, Z, and almost Y operator definitions):

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Note that all three matrices are hermitian (i.e., $\sigma_i^\dagger = \sigma_i$).
a. Show that $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1$.
b. Show that $\sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij}$, where i and j can be x, y, or z.
c. Show that:

$$\begin{aligned} \sigma_x \sigma_y &= i\sigma_z \\ \sigma_y \sigma_z &= i\sigma_x \\ \sigma_z \sigma_x &= i\sigma_y \end{aligned}$$

d. Let $\vec{a}$ and $\vec{b}$ be three dimensional vectors:

$$\vec{a} = (a_x, a_y, a_z), \quad \vec{b} = (b_x, b_y, b_z),$$

and let us define $\vec{\sigma}$ to be an "interesting" vecor whose components are the Pauli matrices:

$$\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z).$$

With these definitions, the dot product of $\vec{a}$ and $sig\vec{m}a$ means:

$$\vec{a} \cdot \vec{\sigma} = a_x \sigma_x + a_y \sigma y + a_z \sigma z$$

Show that the following relation is true (it also implies all the relations you are to have proven above):

$$(\vec{a} \cdot \vec{\sigma})(\vec{b} \cdot \vec{\sigma}) = \vec{a} \cdot \vec{b} + i(\vec{a} \times \vec{b}) \cdot \vec{\sigma}$$

3 Entanglement and Spooky Action at a Distance

Let two q-bits be in the following state:

$$|\Psi\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |10\rangle)$$

a. Is this an entangled state? Why?

Now, let the left-bit belong to Alice, and the right-bit belong to Bob. After the bits are prepared in this state, one bit is sent to Alice, and the other is sent to Bob, and they live three light years away from each other.

b. Show that this state can be expressed as:

$$|\Psi\rangle = \frac{2}{\sqrt{3}}(H_a H_b |00\rangle - \frac{1}{2}|11\rangle)$$

where H_a is the Hadamard operator that acts only on Alice's bit, and H_b is the Hadamard operator that acts only on Bob's bit.

c. Assume Alice and Bob measure their bits directly, without any other action. What is the probability that they will get the results 00, 01, 10 and 11?

d. Since Alice is in posession of her bit, and Bob is in posession of his, they both are able to apply unitary transformations to them before performing a measurement. Assume Alice and Bob are free to apply the Hadamard operator to their own bits before taking a measurement. This generates four cases:

Case A: Apply 1 before measurement. (Alice and Bob do nothing!)

Case B: Apply H_a before measurement. (Alice applies Hadamard operator to her own bit, Bob does nothing.)

Case C: Apply H_b before measurement. (Bob applies Hadamard operator to his own bit, Alice does nothing.)

Case D: Apply $H_a H_b$ before measurement. (Both Alice and Bob apply Hadamard operators to their bits.)

Calculate the probabilities of measuring each result in each case, and fill in the following table. Note that the first row corresponds to part **c** of this problem.

Case	P(00)	P(01)	P(10)	P(11)
A				
B				
C				
D				

e. Assuming there are many two-bit systems of this type, and both Alice and Bob are able to do measurements on them. Is it possible for Alice by any means to determine whether or not Bob applied the Hadamard operator to his bit or not by checking her collected statistics?