Bell's Inequality in Quantum Computing

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1 Introduction

This little paper is an attempt of formulating what is known in quantum mechanics as "Bell's Inequality" in terms of things involved in the discussion of quantum computing only; specifically without invoking knowledge of angular momentum algebra or spin measurements in multiple directions.

We will first discuss measurement in two "ways", and construct a model that does not involve "spooky interactions at a distance". Second, we will consider a system where measurements can be made in three ways by two people, and show that it is not possible to construct a model that does not involve "spooky interactions at a distance".

2 The Definition of the Problem

We have a two-qubit system which is prepared in the following state:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

The two-qubit system, after preparation, is separated to two bits (without performing measurements!). The bit on the left is given to Alice, and the bit on the right is given to Bob. Then, Alice and Bob move to places which are light years away from each other with their bits. Then, each is free to apply any unitary transformations to their bits, and measure them at their leisure.

A few simple observations are in order at this point. First, note that the qubits are in a "maximally entangled" state, that is, the measurement of one of the bits immediately lets us know the outcome of the measurement of the other bit. When one measures either bit, the probability of measuring a zero or a one is equal to $1/2$.

3 Two Kinds of Measurements

Let us assume both Alice and Bob do one of the following things: (a) Do the measurement without any transformations, and (b) do the measurement after applying a Hadamard operator to their bit.

The Hadamard operator is defined by the following two equations:

$$\begin{aligned} H|0\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\ H|1\rangle &= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \end{aligned}$$

Let us examine the possible states when Alice, Bob or both apply the Hadamard operator to their bits:

In case only Alice applies the Hadamard operator:

$$\begin{aligned} (H \otimes 1)|\Psi\rangle &= (H \otimes 1)\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \\ &= \frac{1}{2}(|01\rangle + |11\rangle - |00\rangle + |10\rangle) \\ &= \frac{1}{2}(-|00\rangle + |01\rangle + |10\rangle + |11\rangle \end{aligned}$$

In case only Bob applies the Hadamard operator:

$$\begin{aligned} (1 \otimes H)|\Psi\rangle &= (1 \otimes H)\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \\ &= \frac{1}{2}(|00\rangle - |01\rangle - |10\rangle - |11\rangle) \\ &= -(H \otimes 1)|\Psi\rangle \end{aligned}$$

At this point, note that if one of the two applies the Hadamard operator, the resulting states are identical except for an overall minus sign[1].

In case both parties apply the Hadamard operator:

$$\begin{aligned} (H \otimes H)|\Psi\rangle &= (H \otimes H)\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \\ &= (1 \otimes H)\frac{1}{2}(|01\rangle + |11\rangle - |00\rangle + |10\rangle) \\ &= \frac{1}{2\sqrt{2}}(|00\rangle - |01\rangle + |10\rangle - |11\rangle - |00\rangle - |01\rangle + |10\rangle + |11\rangle) \\ &= \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle) \\ &= -|\Psi\rangle \end{aligned}$$

Similarly here, when both parties apply the Hadamard operator, the resulting state is the same as the initial state, except for an overall minus sign.

[1]An overall phase factor, such as $e^{i\theta}$ does not affect the outcome of any measurement. Thus, as far as measurement results go, these two states are equivalent.

Let us call the situation where nobody applies any operators “1”, the situation where only Alice applies the Hadamard operator “A”, the situation where only Bob applies the Hadamard operator “B”, and the situation where both Bob and Alice apply the Hadamard operator “AB”. Calculating the probabilities of each possible outcome, we can generate the following table:

Situation	P(00)	P(01)	P(10)	P(11)
1	0	$1/2$	$1/2$	0
A	$1/4$	$1/4$	$1/4$	$1/4$
B	$1/4$	$1/4$	$1/4$	$1/4$
AB	0	$1/2$	$1/2$	0

The question we are seeking at this point is this: Can we, by creating a model with “internal parameters”, explain this situation where no interaction happens between the two qubits after they are spatially separated? (Thus obeying Einstein’s locality principle.) Let us try to create such a model:

If we wish to agree that there is to be no more interaction between the two qubits once they are spatially separated, any “decisions” about what a certain measurement’s outcome shall be must be decided by the qubits at the time of separation. We will still agree that once a measurement is done, the state collapses and no more information can be gained; but we will try to make a model for the qubits where they have beforehand decided what to do when any measurement is performed on them.

Under these circumstances, a single qubit has two parameters: The outcome of a direct measurement, and the outcome of a measurement after a Hadamard is applied. For instance, in this model, a qubit can decide “if I am measured directly, I will result in a 1, and if a Hadamard is applied first, and then I am measure, I will result in a 0”. Let us denote such qubits with $(1, 0)$. This makes a total of four types of qubits, $(0, 0)$, $(0, 1)$, $(1, 0)$, and $(1, 1)$. But, we want our model to obey to two-qubit correlations. Thus, if one qubit of the pair is, say, of type $(1, 0)$, the other must be of type $(0, 1)$. In general there is a perfect correspondence between types of qubits.

Since any conclusions we draw will be statistical in nature, and pairs of qubits must decide their types upon separation, assuming N systems of two qubits, there will be the four following types of qubit pairs in the experiment:

Number	Qubit 1	Qubit 2
N_1	$(0, 0)$	$(1, 1)$
N_2	$(0, 1)$	$(1, 0)$
N_3	$(1, 0)$	$(0, 1)$
N_4	$(1, 1)$	$(0, 0)$

where $N_1 + N_2 + N_3 + N_4 = N$.

Here, if we choose $N_1 = N_2 = N_3 = N_4 = N/4$, the situation totally corresponds to the state of affairs in the quantum-mechanical case! If you generate the table of probabilities, you will find the exact table generated by quantum mechanics! So, we have succeeded in creating a hidden-variable model with which we have replicated the results of quantum mechanics without reverting to any spooky interactions at a distance!

4 Three Kinds of Measurements

So far, life has been great. We have been able to replace spooky actions at a distance with a hidden-variable model which obeys Einstein's locality principle. But, we are not limited to taking only two kinds of measurements; we can apply other unitary operators besides the Hadamard operator before taking our measurements. Let us add one more operator to our list[2]; and call it Q. The action of Q is defined as follows:

$$\begin{aligned} Q|0\rangle &= \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle \\ Q|1\rangle &= \frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle \end{aligned}$$

You can easily verify that Q is hermitian ($Q = Q^\dagger$) and unitary ($QQ^\dagger = Q^\dagger Q = Q^2 = 1$). It is just another unitary operator that Alice and Bob may decide to apply to their qubits before taking measurements.

In this new situation, let Alice and Bob choose among three options: Apply 1, H, or Q to their qubits before taking measurements. One important thing is to see the effect of both parties applying the Q operator to their qubits:

$$\begin{aligned} (Q \otimes Q)|\Psi\rangle &= (Q \otimes Q)\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \\ &= (1 \otimes Q)\frac{1}{2\sqrt{2}}(\sqrt{3}|01\rangle + |11\rangle - |00\rangle + \sqrt{3}|10\rangle) \\ &= \frac{1}{4\sqrt{2}}(\sqrt{3}|00\rangle - 3|01\rangle + |10\rangle - \sqrt{3}|11\rangle - \sqrt{3}|00\rangle - |01\rangle + 3|10\rangle + \sqrt{3}|11\rangle) \\ &= \frac{1}{4\sqrt{2}}(4|10\rangle - 4|01\rangle) \\ &= \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle) \\ &= -|\Psi\rangle \end{aligned}$$

Therefore, once again we get back the initial state modified only by a minus sign meaning that any measurements will reveal perfect correlations between the two qubits.

[2] Calling this operator Q is my invention; you will not find it anywhere else in the literature.

This time, we will start by trying to create a hidden-variable model first (we will calculate the quantum-mechanical probabilities later as needed). Since there are three possibilities for each qubit to be measured, and we will again require that no spooky interactions at a distance occur, each qubit must make a decision at the time of separation about *all three possible measurement outcomes*. We will add a third number to our earlier notation to account for measurements after the Q operator is applied first. For instance, a qubit of type $(0, 1, 0)$ will mean a qubit that has decided "I will result in a zero if measured directly, in a 1 if H is applied first, and in a 0 if Q is applied first" at the time of spatial separation. Once again, since when both parties apply the same operator (1, H, or Q) there must be a perfect correlation among the measurements, we have the following eight types of qubit pairs:

Number	Qubit 1	Qubit 2
N_1	$(0,0,0)$	$(1,1,1)$
N_2	$(0,1,0)$	$(1,0,1)$
N_3	$(1,0,0)$	$(0,1,1)$
N_4	$(1,1,0)$	$(0,0,1)$
N_5	$(0,0,1)$	$(1,1,0)$
N_6	$(0,1,1)$	$(1,0,0)$
N_7	$(1,0,1)$	$(0,1,0)$
N_8	$(1,1,1)$	$(0,0,0)$

where $N_1 + N_2 + N_3 + N_4 + N_5 + N_6 + N_7 + N_8 = N$.

If it is possible to obtain the same predictions as quantum mechanics by adjusting these N_i values as necessary, we will have successfully developed a local, hidden-variable model that does away with spooky actions at a distance.

What is the probability that Alice applied no transformations and got the result 1 in her measurement, and Bob applied H and also got 1 in his measurement? Denoting this event as $(1_A \to 1; H_B \to 1)$, we have:

$$P(1_A \to 1; H_B \to 1) = \frac{N_3 + N_7}{N}$$

In similar notation, we have:

$$\begin{aligned} P(1_A \to 1; Q_B \to 1) &= \frac{N_3 + N_4}{N} \\ P(Q_A \to 1; H_B \to 1) &= \frac{N_7 + N_5}{N} \end{aligned}$$

Now, since all N_i are greater than or equal to zero, we have:

$$N_3 + N_7 \leq (N_3 + N_4) + (N_7 + N_5)$$

After dividing through by N, this yields the following inequality:

$$P(1_A \to 1; H_B \to 1) \leq P(1_A \to 1; Q_B \to 1) + P(Q_A \to 1; H_B \to 1)$$

This is the celebrated Bell's inequality. The important thing is, this holds for any hidden-variable, local model *no matter how the different N_i values are assigned*. So, if this inequality is violated, it means *there is no hidden-variable, local interaction model that can be constructed* and therefore *spooky interactions at a distance exist*.

Now, let us perform the necessary quantum mechanical calculations for our system to calculate the probabilities in Bell's inequality, and let us see whether they satisfy the inequality or not.

First case is, Alice applying nothing, and Bob applying the Hadamard. Thus the state is (which we have already calculated earlier:

$$(1 \otimes H)|\Psi\rangle = \frac{1}{2}(|00\rangle - |01\rangle - |10\rangle - |11\rangle)$$

therefore

$$P(1_A \to 1; H_B \to 1) = \frac{1}{4}$$

Second case is, Alice applying nothing, and Bob applying Q (this one we have not calculated before):

$$\begin{aligned}(1 \otimes Q)|\Psi\rangle &= (1 \otimes Q)\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \\ &= \frac{1}{\sqrt{8}}(|00\rangle - \sqrt{3}|01\rangle - \sqrt{3}|10\rangle - |11\rangle)\end{aligned}$$

therefore

$$P(1_A \to 1; Q_B \to 1) = \frac{1}{8}$$

The third and final case is, Alice applying Q, and Bob applying the Hadamard (which is the most complicated case, it seems):

$$\begin{aligned}(Q \otimes H)|\Psi\rangle &= (Q \otimes H)\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \\ &= (1 \otimes H)\frac{1}{2\sqrt{2}}(\sqrt{3}|01\rangle + |11\rangle - |00\rangle + \sqrt{3}|10\rangle) \\ &= \frac{1}{4}(\sqrt{3}|00\rangle - \sqrt{3}|01\rangle + |10\rangle - |11\rangle - |00\rangle - |01\rangle + \sqrt{3}|10\rangle + \sqrt{3}|11\rangle) \\ &= \frac{1}{4}\left[(\sqrt{3}-1)|00\rangle - (\sqrt{3}+1)|01\rangle + (\sqrt{3}+1)|10\rangle + (\sqrt{3}-1)|11\rangle\right]\end{aligned}$$

therefore

$$\begin{aligned}
P(Q_A \to 1; H_B \to 1) &= \left(\frac{\sqrt{3}-1}{4}\right)^2 \\
&= \frac{4-2\sqrt{3}}{16} \\
&= 0.03349365
\end{aligned}$$

Let us plug these values in to Bell's inequality:

$$\begin{aligned}
P(1_A \to 1; H_B \to 1) &\leq P(1_A \to 1; Q_B \to 1) + P(Q_A \to 1; H_B \to 1) \\
\frac{1}{4} &\leq \frac{1}{8} + \left(\frac{\sqrt{3}-1}{4}\right)^2 \\
0.25 &\leq 0.125 + 0.03349365 \\
0.25 &\leq 0.15849365
\end{aligned}$$

The predictions of quantum mechanics definitely contradict *any* hidden-variable model than can be constructed. Thus, there are two possibilities:

- Quantum mechanics is wrong, and there are hidden-variable local-interaction models that can explain these natural phenomena. There is no such thing as a spooky-action at a distance.
- Quantum mechanics is correct, and it is not possible to construct any hidden-variable local-interaction models whatsoever. Spooky actions at a distance exist in nature.

There is only one way to choose between the two possibilities: Experiment. Experiments on this kind of correlations has been done, and Bell's inequality was found to be violated with great certainty. Thus, we conclude:

- However disturbing it might seem, quantum mechanics is correct to the best of our knowledge, and spooky actions at a distance do occur.
- It is not possible to construct any hidden-variable models in order to obey the principle of locality. Nature does work in non-local ways.
- Despite this interaction, it is still not possible to *transmit any information* over this kind of interaction. This is why it is called a *spooky interaction* rather than a *real interaction*.