CSE 505 Problem Set 3

Date: Thursday, October 28^{th}, 2004
Due date: Thursday, November 4^{th}, 2004

• **Problem 1** : Gaussian Integrals

a. The "basic" Gaussian integral is:

$$\int_{-\infty}^{+\infty} e^{-x^2} dx$$

Show that the value of this integral is $\sqrt{\pi}$. This can be done as follows. Let $I = \int_{-\infty}^{+\infty} e^{-x^2} dx$. Then,

$$I^2 = \int_{-\infty}^{+\infty} e^{-x^2} dx \cdot \int_{-\infty}^{+\infty} e^{-y^2} dy$$

and

$$I^2 = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-(x^2+y^2)} dx dy$$

Consider this to be an integral over all $x - y$ plane. Now, change to polar coordinates, and do a suitable change of variables to reach the required result.

b. Show that

$$\int_{-\infty}^{+\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$$

c. Show that

$$\int_{-\infty}^{+\infty} e^{-a(x-b)^2} dx = \sqrt{\frac{\pi}{a}}$$

d. Finally, show that

$$\int_{-\infty}^{+\infty} e^{-(ax^2+bx+c)} dx = \sqrt{\frac{\pi}{a}} \cdot e^{\frac{b^2-4ac}{4a}}$$

Hint: Complete the square in the exponent.

• **Problem 2**

A continuous random variable x is said to be a *Gaussian random variable* if it is distributed as:

$$f_x(x_0) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \cdot e^{-\frac{(x_0-\overline{x})^2}{2\sigma_x^2}}$$

It can be shown that the expectation and variance of the random variable x in this case are $\overline{x}$ and σ_x^2, respectively. It can also be shown that this PDF is properly normalized.

For this problem, assume that x is a zero-mean ($\overline{x} = 0$) Gaussian random variable with variance σ^2 ($\sigma_x^2 = \sigma^2$). Derive an expression for $E[x^n]$, the nth moment, that is valid for all integer $n \geq 0$.

Hint: Consider the two cases of n odd, and n even.

• **Problem 3**

To do this problem, you might find the results of problem 3 useful.

A *central Chi-Square* random variable y, with n degrees of freedom, is defined as:

$$y = x_1^2 + x_2^2 + \cdots + x_n^2$$

where the x_i's are independent and identically distributed zero-mean Gaussian random variables with variance σ^2. Find the expectation and variance of y.