CSE 505 Final Examination

Date: Wednesday, January 5^{th}, 2005
Duration: One Week

• **Problem 1**

A factory has a conveyor belt, on which wooden planks of different sizes travel with no gaps in between. The lengths of the planks are exponentially distributed, with parameter λ. (In other words, if we denote the length of a single plank with the random variable x, $f_x(x_0) = \lambda e^{-\lambda x_0}$.)

Given this information, the average length of the planks is $1/\lambda$.

Here is an interesting apparent paradox: Since the lengths are exponentially distributed, the arrivals of the endpoints (separation points) is a Poisson process. Suppose while the planks are passing, we stop the belt at a random time, say, by driving a knife into the plank currently in front of us. Now, the expectation value of the length until the end of the plank (from our knife) is $1/\lambda$. Now, if we consider the distance *since* the beginning of the plank to our knife, the expectation value of that is also $1/\lambda$. Therefore, the expectation value of the total length of the plank, which is the sum of these two lengths, is $2/\lambda$!

The question is this: The average length of the planks is $1/\lambda$, but when we catch one, we see the average length is $2/\lambda$! How does this happen? Which one is right? Or what is happening? Only explanations hitting the problem on the head will receive points.

• **Problem 2**

The state transition matrix for a discrete-state, discrete transition Markov process with two states is given as:

$$A = \begin{bmatrix} a & b \\ 1-a & 1-b \end{bmatrix}$$

In this notation, the state probability vector is denoted as a column vector (which is different than what we have in the book). Note that this is the most general case for a two-state system.

Since this A is the one-step transition matrix, the two, three, n-step transition matrices are A^2, A^3, A^n.

The most general state probability vector is:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where $x_1 + x_2 = 1$.

a. Calculate the two- and three- step state transition matrices.

b. By any means you can, calculate the elements of the matrix A^n as a function of n, for all $n > 0$.

c. Define $\vec{x_n} = A^n\vec{x}$. Calculate $\vec{x}_\infty = \lim_{n\to\infty} \vec{x_n}$, which is the limiting state probability vector.

d. Under what conditions does $\vec{x}_\infty$ become independent of $\vec{x}$, the initial state probability vector?

e. How fast does $\vec{x_n}$ converge to $\vec{x}_\infty$? In other words, what is the dependence of convergence on the parameters a and b?

• **Problem 3**

a. A discrete-state discrete-transition Markov process has three states. The state transition probabilities at each step from state i to state j are given by the state transition matrix p_{ij}. It is known that $p_{ij} = p_{ji} > 0$ for all pairs (i, j). Find the limiting state probabilities P_i in terms of p_{ij}.

b. This time, consider a Markov process with N states, with the same property that $p_{ij} = p_{ji} > 0$. Find the limiting state probabilities P_i in terms of p_{ij}. (The result is very important for quantum statistical mechanics.)